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Matrix polynomials with spectral radius equal to the numerical radius

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ABSTRACT

In this paper we study a class of matrix polynomials with the property that spectral radius and numerical radius coincide. Special attention is paid to the spectrum on the boundary of the numerical range.

1. Introduction

Let the matrix polynomial $B(z) = B_m z^m - \sum_{j=0}^{m-1} B_j z^j \in \mathbb{C}^{n\times n}[z]$ be nonsingular (i.e. with $\det B(z)$ not identically zero). The *numerical range* of $B(z)$ is the set

$$W(B) = \{\lambda \in \mathbb{C};\ y^* B(\lambda) y = 0 \text{ for some } y \in \mathbb{C}^n,\ y \neq 0\}.$$

The first systematic study of the numerical range of matrix polynomials is due to Li and Rodman [6]. For extensions to operator polynomials in Hilbert space we refer to [15]. Geometric properties of $W(B)$ are gathered together in [13]. The numerical range is a tool to determine the stability radius [11] or to obtain factorizations [8] of a matrix polynomial. In [9] and [12] it is used to study Perron matrix polynomials. Let

$$\sigma(B) = \{\lambda \in \mathbb{C};\ \det B(\lambda) = 0\}$$

be the *spectrum* of $B(z)$. In accordance with [1] we call λ a *characteristic value* of $B(z)$ if $\lambda \in \sigma(B)$. Because of $\sigma(B) \subseteq W(B)$ the *spectral radius*

$$r(B) = \max\{|\lambda|;\ \lambda \in \sigma(B)\} \tag{1.1}$$

and the *numerical radius*

$$w(B) = \sup\{|\lambda|;\ \lambda \in W(B)\}$$

of $B(z)$ satisfy $r(B) \leqslant w(B)$. In this note we study a class of matrix polynomials which satisfy $r(B) = w(B)$. Characteristic values on the boundary of the numerical range and the corresponding elementary divisors will be investigated in detail. (The

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suggested reference on elementary divisors of matrix polynomials is [3].) We say that $\lambda \in \sigma(B)$ is a *normal* characteristic value of $B(z)$ if $\operatorname{Ker} B(\lambda)^* = \operatorname{Ker} B(\lambda)$, and we call λ *semisimple* if the corresponding elementary divisors are linear. Let $A \in \mathbb{C}^{n\times n}$ be normal, i.e. $AA^* = AA^*$, and let $|A| = (AA^*)^{1/2}$ be the *absolute value* of A. To a monic matrix polynomial $B(z) = Iz^m - \sum_{j=0}^{m-1} B_j z^j$ with normal coefficients B_j we associate the matrix polynomial

$$\bar{\bar{B}}(z) = Iz^m - \sum_{j=0}^{m-1} |B_j| z^j. \tag{1.2}$$

The main result of this paper is the following.

Theorem 1.1. *Let $B(z) = Iz^m - \sum_{j=0}^{m-1} B_j z^j$ be a matrix polynomial with normal coefficients B_j and suppose $r(B) = r(\bar{\bar{B}})$. Then the following hold*:

(i) $w(B) = r(B)$.
(ii) *If $\lambda \in \sigma(B)$ and $|\lambda| = r(B)$, then λ is normal and semisimple.*

The proof of the theorem will be given in Section 2.4. It is based on results in Sections 2.1–2.3. In Section 2.1 we deal with matrix polynomials with semidefinite coefficients, in Section 2.2 we derive a matrix Cauchy bound, and in Section 2.3 we are concerned with characteristic values on the boundary of the numerical range.

Notation. Let R, S be hermitian matrices. We write $R \geqslant 0$ if R is positive semidefinite and $R \geqslant S$ if $R - S \geqslant 0$. Similarly, $R \leqslant S$ means that $R - S$ is negative semidefinite.

2. Numerical radius and spectral radius

2.1. Positive semidefinite coefficients

In this section we assume that $B(z) \in \mathbb{C}^{n\times n}[z]$ is monic,

$$B(z) = Iz^m - \big(B_{m-1}z^{m-1} + \cdots + B_1 z + B_0\big). \tag{2.1}$$

Let $v \in \mathbb{C}^n$ with $v^*v = 1$, and define

$$b_j^{(v)} = v^* B_j v, \quad j = 0, \ldots, m-1, \qquad b^{(v)}(z) = v^* B(z) v, \tag{2.2}$$

such that $b^{(v)}(z) = z^m - \sum b_j^{(v)} z^j$.

For the proof of Theorem 2.3 below we need two lemmas on polynomials. We use the notation (1.1). Thus, if $b(z)$ is a complex polynomial, then $r(b) = \max\{|\lambda|;\, b(\lambda) = 0\}$.

Lemma 2.1. *(See [14, p. 243].) Let*

$$b(z) = z^m - \big(b_{m-1}z^{m-1} + \cdots + b_1 z + b_0\big)$$

be a complex polynomial, $b(z) \neq z^m$. Set

$$\bar{\bar{b}}(z) = z^m - \big(|b_{m-1}|z^{m-1} + \cdots + |b_1|z + |b_0|\big).$$

Then $\bar{\bar{b}}(z)$ has a unique real positive root ρ, and $r(b) \leqslant r(\bar{\bar{b}}) = \rho$.

Lemma 2.2. *(See [14, p. 243], [10, p. 3].) Let*

$$b(z) = z^m - \big(b_{m-1}z^{m-1} + \cdots + b_1 z + b_0\big)$$

be a real polynomial, $b(z) \neq z^m$. Suppose $b_j \geqslant 0$, $j = 0, \ldots, m-1$.

(i) *Then $b(z)$ has exactly one positive zero ρ. Moreover ρ is a continuous and increasing function of each $b_j \in [0, \infty)$, $j = 0, \ldots, m-1$.*
(ii) *If*

$$\sum_{j=0}^{m-1} b_j \leqslant 1$$

then $r(b) \leqslant 1$, and the zeros of $b(z)$ on the unit circle (if any) are simple.

The following theorem deals with matrix polynomials (2.1) with positive semidefinite coefficients B_j. It is a special case of results on operator polynomials in Hilbert space [15].

Theorem 2.3. *Suppose that the coefficients B_j, $j = 0, \dots, m-1$, of the matrix polynomial* (2.1) *are positive semidefinite. Then the following hold*:

(i) $w(B) = r(B)$.
(ii) *The condition $w(B) \leqslant 1$ is equivalent to*

$$\sum_{j=0}^{m-1} B_j \leqslant I. \tag{2.3}$$

Proof. Let $\rho = w(B)$. If $\rho = 0$, then clearly $B(z) = Iz^m$, and there is nothing to prove. Assume $\rho > 0$. Then $B(z) \neq Iz^m$ and $\sum_{j=0}^{m-1} B_j \neq 0$.

(i) Since $W(B)$ is closed we have $\rho = |\lambda|$ for some $\lambda \in W(B)$. Consider a corresponding vector $v \in \mathbb{C}^n$ with

$$v^* B(\lambda) v = 0 \quad \text{and} \quad v^* v = 1. \tag{2.4}$$

Then $b^{(v)}(\lambda) = 0$, and $b_j^{(v)} \geqslant 0$, $j = 0, \dots, m-1$, and $\sum b_j^{(v)} > 0$. Hence (by Lemma 2.2) there exists a unique positive root $\hat{\rho}$ of $b^{(v)}(z)$. Moreover, Lemma 2.1 implies $r(b^{(v)}) = \hat{\rho}$. Thus $\rho \leqslant \hat{\rho}$. Because of $b^{(v)}(\hat{\rho}) = 0$ we have $\hat{\rho} \in W(B)$, and therefore $\hat{\rho} \leqslant \rho$. Hence $\hat{\rho} = \rho$, and therefore $\rho \in W(B)$. Suppose $y^* B(\rho) y < 0$ for some $y \neq 0$. If $t \in \mathbb{R}_>$ is sufficiently large then $y^* B(t) y > 0$. Hence $y^* B(s) y = 0$ for some $s > \rho$, and we would have $w(B) > \rho$. Therefore we obtain $B(\rho) \geqslant 0$. Then $v^* B(\rho) v = 0$ implies $B(\rho) v = 0$. Hence $\rho \in \sigma(B)$, and $\rho \leqslant r(B)$. Then $r(B) \leqslant w(B)$ yields $r(B) = w(B)$.

(ii) Suppose $0 < \rho = w(B) \leqslant 1$. We have seen that $B(\rho) \geqslant 0$. Therefore $\rho^m I \geqslant \sum_{j=0}^{m-1} B_j \rho^j$ implies

$$I \geqslant \sum_{j=0}^{m-1} B_j \rho^{j-m} \geqslant \sum_{j=0}^{m-1} B_j.$$

This proves (2.3). Let $\lambda \in W(B)$ and let v be a corresponding vector satisfying (2.4), and let $b^{(v)}(z)$ be the polynomial in (2.2). Then (2.3) implies $\sum_{j=0}^{m-1} b_j^{(v)} \leqslant 1$. Hence Lemma 2.2(i) yields $|\lambda| \leqslant 1$, and therefore $w(B) \leqslant 1$. $\square$

Corollary 2.4. *Let $B(z)$ be a matrix polynomial as in the previous theorem. We have $w(B) = 1$ if and only if*

$$B(1) \geqslant 0 \quad \textit{and} \quad \operatorname{Ker}\big(B(1)\big) \neq 0 \tag{2.5}$$

or equivalently, if and only if

$$\sum_{j=0}^{m-1} B_j \leqslant I \quad \textit{and} \quad \operatorname{Ker}\left(I - \sum_{j=0}^{m-1} B_j\right) \neq 0.$$

Proof. We have seen in the proof of Theorem 2.3(ii) that $w(B) = 1$ implies (2.5). Conversely, if (2.5) holds, then it follows from $\operatorname{Ker}(B(1)) \neq 0$ that $1 \in \sigma(B)$. Therefore $1 \leqslant r(B)$. From $B(1) \geqslant 0$ we obtain $w(B) \leqslant 1$. Hence $w(B) \leqslant 1 \leqslant r(B)$ yields $w(B) = 1$. $\square$

2.2. Normal coefficients

Lemma 2.5. *If $A \in \mathbb{C}^{n \times n}$ is normal then*

$$\left| v^* A v \right| \leqslant v^* |A| v \quad \textit{for all } v \in \mathbb{C}^n. \tag{2.6}$$

Proof. A normal matrix is unitarily similar to a diagonal matrix. Hence it suffices to prove (2.6) for $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. In this case we have $|A| = \operatorname{diag}(|\lambda_1|, \dots, |\lambda_n|)$, and (2.6) is obvious. $\square$

If A is not normal, then the *left absolute value* $|A|_\ell = (AA^*)^{1/2}$ and the *right absolute value* $|A|_r = (A^*A)^{1/2}$ do not coincide (see also [2, Condition 71]). Note that (2.6) is not valid if the absolute value $|A|$ is replaced by $|A|_\ell$ or $|A|_r$.

Example 2.6. Consider $A = PU \in \mathbb{R}^{2\times 2}$,

$$P = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} > 0, \quad a > b > 0, \qquad U = \begin{pmatrix} \cos 2\tau & \sin 2\tau \\ -\sin 2\tau & \cos 2\tau \end{pmatrix}, \quad \pi/4 > \tau > 0.$$

The matrix A is not normal because of $PU \neq UP$. We have

$$|A|_\ell = \left(AA^*\right)^{1/2} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

and

$$|A|_r = \left(A^*A\right)^{1/2} = \begin{pmatrix} a\cos^2 2\tau + b\sin^2 2\tau & (a-b)\cos 2\tau \sin 2\tau \\ (a-b)\cos 2\tau \sin 2\tau & a\sin^2 2\tau + b\cos^2 2\tau \end{pmatrix}.$$

Let $v = \binom{1}{1}$. We shall see that $|v^*Av| > v^*|A|_\ell$ if τ is sufficiently small. From $v^*Av = (a+b)\cos 2\tau + (a-b)\sin 2\tau = |v^*Av|$ and $v^*|A|_\ell v = a+b$ follows

$$\begin{aligned} \left|v^*Av\right| - v^*|A|_\ell v &= (a+b)(\cos 2\tau - 1) + (a-b)\sin 2\tau = (a+b)\left(-2\sin^2\tau\right) + (a-b)2\sin\tau\cos\tau \\ &= 2\sin^2\tau\left[-(a+b) + (a-b)\cot\tau\right]. \end{aligned}$$

Hence $|v^*Av| > v^*|A|_\ell v$ if and only if $\cot\tau > \frac{a+b}{a-b}$.

We extend Lemma 2.1 to matrix polynomials.

Theorem 2.7. *Let $B(z) = Iz^m - \sum_{j=0}^{m-1} B_j z^j$ be a matrix polynomial with normal coefficients B_j, and let $\bar{\bar{B}}(z)$ be defined as in* (1.2). *Then*

$$r(B) \leqslant w(B) \leqslant w(\bar{\bar{B}}) = r(\bar{\bar{B}}). \tag{2.7}$$

Proof. Let $v \in \mathbb{C}^n$, $v^*v = 1$. In addition to (2.2) we define

$$\bar{\bar{b}}^{(v)}(z) = z^m - \sum \left|b_j^{(v)}\right| z^j.$$

Then Lemma 2.1 implies $r(b^{(v)}) \leqslant r(\bar{\bar{b}}^{(v)})$. To $\bar{\bar{B}}(z)$ we associate the polynomial

$$a^{(v)}(z) = v^*\bar{\bar{B}}(z)v = z^m - \sum v^*|B_j|vz^j.$$

Because of (2.6) the coefficients of $\bar{\bar{b}}^{(v)}(z)$ and $a^{(v)}(z)$ are related by

$$\left|b_j^{(v)}\right| = \left|v^*B_jv\right| \leqslant v^*|B_j|v, \quad j = 0, \ldots, m-1.$$

Therefore Lemma 2.2(i) implies $r(\bar{\bar{b}}^{(v)}) \leqslant r(a^{(v)})$. Then

$$\begin{aligned} w(B) &= \sup\left\{|\lambda|;\ v^*B(\lambda)v = 0 \quad \text{for some } v \in \mathbb{C}^n,\ v \neq 0\right\} = \sup\left\{r\left(b^{(v)}\right);\ v \in \mathbb{C}^n,\ v^*v = 1\right\} \\ &\leqslant \sup\left\{r\left(\bar{\bar{b}}^{(v)}\right);\ v \in \mathbb{C}^n,\ v^*v = 1\right\} \leqslant \sup\left\{r\left(a^{(v)}\right);\ v \in \mathbb{C}^n,\ v^*v = 1\right\} = w(\bar{\bar{B}}). \end{aligned}$$

From Theorem 2.3 we obtain $w(\bar{\bar{B}}) = r(\bar{\bar{B}})$. Therefore we have (2.7). □

Let μ be a positive real number. Set

$$B_\mu(z) = Iz^m - \sum_{j=0}^{m-1} B_j \mu^{-(m-j)} z^j.$$

Then $\mu^{-m}B(z\mu) = B_\mu(z)$ and $r(B_\mu) = \frac{1}{\mu} r(B)$. Similarly, $w(B_\mu) = \frac{1}{\mu} w(B)$, and $r(\bar{\bar{B}}_\mu) = \frac{1}{\mu} r(\bar{\bar{B}})$. So, whenever convenient we may assume $r(\bar{\bar{B}}) = 1$.

The condition $r(B) = r(\bar{\bar{B}})$ is essential for Theorem 1.1. Therefore let us consider the case where $B(z) = b(z)$ is a polynomial, and then make an observation on the general case.

Proposition 2.8. *Let $b(z) = z^m - \sum_{j=0}^{m-1} b_j z^j \in \mathbb{C}[z]$. Set $\bar{\bar{b}}(z) = z^m - \sum_{j=0}^{m-1} |b_j| z^j$. Then $r(b) = r(\bar{\bar{b}})$ if and only if*

$$b(z) = \lambda^m \bar{\bar{b}}\big(\lambda^{-1} z\big) \quad \textit{for some } \lambda = e^{i\phi}.$$

Proof. We assume $r(\bar{\bar{b}}) = 1$, i.e. $\sum_{j=0}^{m-1} |b_j| = 1$. Suppose $b(\lambda) = 0$ and $|\lambda| = r(\bar{\bar{b}}) = 1$. Then $\lambda^m = \sum_{j=0}^{m-1} b_j \lambda^j$, and

$$1 = \sum_{j=0}^{m-1} b_j \lambda^{j-m} = \sum_{j=0}^{m-1} \big|b_j \lambda^{j-m}\big| = \sum_{j=0}^{m-1} |b_j| = 1.$$

Hence $b_j \lambda^{j-m} = |b_j|$, $j = 0, \ldots, m-1$, and therefore

$$b(z) = z^m - \big(\lambda |b_{m-1}| z^{m-1} + \cdots + \lambda^{m-1} |b_1| z + \lambda^m |b_0|\big) = \lambda^m \bar{\bar{b}}\big(\lambda^{-1} z\big). \qquad \square$$

Proposition 2.9. *Let $B(z) = I z^m - \sum_{j=0}^{m-1} B_j z^j$ be a matrix polynomial with normal coefficients. Let $r(\bar{\bar{B}}) = \mu$ and let the columns of $V = (v_1, \ldots, v_k)$ be a basis of $\operatorname{Ker} \bar{\bar{B}}(\mu)$. Then*

$$r(B) = r(\bar{\bar{B}}) \tag{2.8}$$

if and only if

$$\det V^* B(\lambda) V = 0 \quad \textit{for some } \lambda \textit{ with } |\lambda| = r(\bar{\bar{B}}), \tag{2.9}$$

or equivalently if and only if

$$r\big(V^* B(z) V\big) = r(\bar{\bar{B}}). \tag{2.10}$$

Proof. We may assume $\mu = r(\bar{\bar{B}}) = 1$. We can also assume that the columns of V are an orthonormal system such that $V^* V = I_k$. It is obvious that $\sigma(V^* B V) \subseteq \sigma(B)$. Hence $r(V^* B V) \leqslant r(B)$.

(2.8) $\Rightarrow$ (2.9). Suppose $r(B) = r(\bar{\bar{B}}) = 1$. Let $\lambda \in \sigma(B)$, $|\lambda| = 1$, and

$$B(\lambda) v = 0, \quad v^* v = 1. \tag{2.11}$$

Using the notation and the arguments of the proof of Theorem 2.7 we note that

$$r(B) = r\big(b^{(v)}\big) \leqslant r\big(a^{(v)}\big) \leqslant r(\bar{\bar{B}}).$$

Hence $r(a^{(v)}) = 1$, and therefore $v^* \bar{\bar{B}}(1) v = 0$. Recall that the coefficients $|B_j|$ of $\bar{\bar{B}}(z)$ are positive semidefinite matrices. Hence $\bar{\bar{B}}(1) \geqslant 0$ implies $\bar{\bar{B}}(1) v = 0$, that is $v \in \operatorname{Ker} \bar{\bar{B}}(1)$. Therefore $v = V y$, $y^* y = 1$. Then (2.11) yields $V^* B(\lambda) V y = 0$, and we obtain (2.9).

(2.9) $\Rightarrow$ (2.10). From (2.9) follows $r(V^* B V) \geqslant r(\bar{\bar{B}})$. Thus $r(V^* B V) \leqslant r(B)$ yields $r(B) = r(\bar{\bar{B}})$. The implication (2.10) $\Rightarrow$ (2.9) is obvious.

(2.9) $\Rightarrow$ (2.8). Suppose (2.9) is satisfied. Then $B(\lambda) V y = 0$ for some λ, $|\lambda| = r(\bar{\bar{B}})$, and $y \in \mathbb{C}^k$, $y^* y = 1$. Hence $r(B) \geqslant r(\bar{\bar{B}})$. Then $r(B) \leqslant r(\bar{\bar{B}})$ yields (2.8). $\square$

The next lemma is a technical result, which will be needed to prove semisimplicity of characteristic values.

Lemma 2.10. *Let $B(z) = I z^m - \sum_{j=0}^{m-1} B_j z^j$ have normal coefficients B_j and let $r(B) = r(\bar{\bar{B}})$. If $|\lambda| = r(\bar{\bar{B}})$ and $v \neq 0$ then $v^* B'(\lambda) v \neq 0$.*

Proof. We can assume $r(\bar{\bar{B}}) = 1$ and $v^* v = 1$. Then Theorem 2.3(ii) implies

$$v^* \left(\sum_{j=0}^{m-1} |B_j| \right) v \leqslant 1. \tag{2.12}$$

Suppose $v^* B'(\lambda) v = 0$ and $v^* v = 1$. Then

$$m\lambda^{m-1} = v^* \left(\sum_{j=0}^{m-1} j\lambda^{j-1} B_j \right) v$$

implies

$$1 = \sum_{j=0}^{m-1} \frac{j}{m} \frac{1}{\lambda^{m-j}} v^* B_j v.$$

Hence, taking (2.12) into account, we obtain

$$1 \leqslant \sum_{j=0}^{m-1} \frac{j}{m} \left| v^* B_j v \right| < \sum_{j=0}^{m-1} v^* |B_j| v \leqslant 1,$$

which is a contradiction. □

2.3. Characteristic values on $\partial W(B)$

The *field of values* (or numerical range) of a matrix $A \in \mathbb{C}^{n \times n}$ is the set $F(A) = \{v^* A v;\ v \in \mathbb{C}^{n \times n},\ v^* v = 1\}$. If $B(z) = zI - A$ then $F(A) = W(B)$. We recall two results on characteristic values on the boundary of the numerical range.

Theorem 2.11. *(See [7, p. 103].) If $\lambda \in \partial W(B)$ then $0 \in \partial F(B(\lambda))$.*

Theorem 2.12. *(See [4, p. 51], [5, p. 235].) Let $A \in \mathbb{C}^{n \times n}$. If $\lambda \in \partial F(A) \cap \sigma(A)$ then*

$$\operatorname{Ker}(\lambda I - A) = \operatorname{Ker}(\lambda I - A)^*.$$

We remark that $\lambda \in \sigma(B)$ is equivalent to $0 \in \sigma(B(\lambda))$. Thus, combining the preceding theorems we obtain the following.

Theorem 2.13. *If $\lambda \in \partial W(B) \cap \sigma(B)$ then λ is a normal characteristic value, i.e.*

$$\operatorname{Ker} B(\lambda) = \operatorname{Ker} B(\lambda)^*. \tag{2.13}$$

Proof. If $\lambda \in \sigma(B)$ lies on the boundary of $W(B)$ then $0 \in \partial F(B(\lambda))$. Hence Theorem 2.12 implies (2.13). □

2.4. Proof of the main theorem

We recall (see e.g. [3]) that a characteristic value λ of $B(z)$ is semisimple if and only if all corresponding Jordan chains have length 1. That means, if $v \in \operatorname{Ker} B(\lambda)$ and $v \neq 0$, then there does not exist a vector $w \in \mathbb{C}^n$ such that

$$B'(\lambda)v + B(\lambda)w = 0. \tag{2.14}$$

Proof of Theorem 1.1.

(i) If $r(B) = r(\bar{\bar{B}})$ then $w(B) = r(B)$ follows immediately from (2.7).
(ii) Let $r(\bar{\bar{B}}) = 1$. Then the condition $r(B) = r(\bar{\bar{B}})$ implies $w(B) = 1$. Hence, if $\lambda \in \sigma(B)$ and $|\lambda| = r(\bar{\bar{B}}) = 1$ then $\lambda \in \partial W(B) \cap \sigma(B)$, and by Theorem 2.13 we have $\operatorname{Ker} B(\lambda) = \operatorname{Ker} B(\lambda)^*$. The characteristic value λ is semisimple. Otherwise there exist vectors $v, w \in \mathbb{C}^n$ which satisfy $B(\lambda)v = 0$, $v \neq 0$, and (2.14). Then $v^* B'(\lambda) v = 0$, in contradiction to Lemma 2.10. □

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